Resit Exam — Analysis (WPMA14004)

Tuesday 10 July 2018, 9.00h–12.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (3 + 6 + 6 = 15 points)

Consider the set $A = \left\{ \dfrac{n}{n+k} \; : \; n, k \in \mathbb{N} \right\}$.

(a) State the Axiom of Completeness.

(b) Prove that $\sup A = 1$.

(c) Prove that $\inf A = 0$.

Problem 2 (5 + 5 + 5 = 15 points)

Let $x_1 = 5$, and define the sequence

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n} \right).$$

Prove the following statements:

(a) $x_n > \sqrt{5}$ for all $n \in \mathbb{N}$. Hint: first show by induction that $x_n^2 > 5$.

(b) $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. Hint: compute the difference $x_n - x_{n+1}$.

(c) $x = \lim x_n$ exists and $x = \sqrt{5}$.

Problem 3 (3 + 9 + 3 = 15 points)

(a) Give the definition of a limit point of a set.

(b) Recall that the closure of a set A is given by $\overline{A} = A \cup L$, where L is the set of limit points of A.

Prove the following statement:

$$x \text{ is a limit point of } A \quad \Leftrightarrow \quad x \text{ is a limit point of } \overline{A}.$$

(c) Show that for any set A the closure $\overline{A}$ is closed.

Problem 4 (5 + 5 + 5 = 15 points)

Let $f : [0, 3] \to \mathbb{R}$ be differentiable, and assume that

$$f(0) = 1, \quad f(1) = 2, \quad f(3) = 2.$$

Prove the following statements:

(a) There exists $a \in [0, 3]$ such that $f(a) = a$;

(b) There exists $b \in [0, 3]$ such that $f'(b) = 1/3$;

(c) There exists $c \in [0, 3]$ such that $f'(c) = 1/4$.

Problem 5 (9 + 6 = 15 points)

Assume that the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

converges at some point $x_0 > 0$.

(a) Show that the series $\sum_{n=0}^{\infty} |a_n x^n|$ converges for any $x \in (-x_0, x_0)$.
Hint: show that the sequence $(a_n x_0^n)$ is bounded and then apply the comparison test.

(b) Let $0 < c < x_0$. Prove that $\sum_{n=0}^{\infty} a_n x^n$ converges *uniformly* on the interval $[-c, c]$.
Hint: use the Weierstrass test.

Problem 6 (10 + 5 = 15 points)

Let $\{r_1, r_2, r_3, \dots\}$ be an enumeration of all rational numbers in $[0, 1]$, and define

$$f_j(x) = \begin{cases} 1 & \text{if } x = r_j \text{ for some } j \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Prove that f_j is integrable on $[0, 1]$ for all $j \in \mathbb{N}$ and compute $\int_0^1 f_j$.

(b) Is $F(x) = \sum_{j=1}^{\infty} f_j(x)$ integrable on $[0, 1]$?

End of test (90 points)

Solution of problem 1 (3 + 6 + 6 = 15 points)

(a) Every set of real numbers that is bounded above has a least upper bound.
(3 points)

(b) We claim that $\sup A = 1$. First note that the number 1 is an upper bound of A. Indeed,

$$n + k > n \quad \Rightarrow \quad \frac{n}{n+k} < \frac{n}{n} = 1.$$

(1 point)

To show that we have found the least upper bound we can proceed in two different ways.

Method 1. Let u be any upper bound for A, then $n/(n+k) \leq u$ for all $n, k \in \mathbb{N}$. In particular, by taking $k = 1$, it follows that $n/(n+1) \leq u$ for all $n \in \mathbb{N}$. The sequence $x_n = n/(n+1)$ is convergent. By the Order Limit Theorem it follows that $\lim x_n \leq u$ which implies that $1 \leq u$. This shows that $\sup A = 1$.
(5 points)

Method 2. Let $\epsilon > 0$ be arbitrary, and take $k = 1$ and $n \in \mathbb{N}$ such that $n > (1 - \epsilon)/\epsilon$. Then

$$\frac{n}{n+k} = \frac{n}{n+1} = \frac{1}{1 + 1/n} > 1 - \epsilon.$$

This shows that $1 - \epsilon$ is not an upper bound of A. We conclude that $\sup A = 1$.
(5 points)

(c) We claim that $\inf A = 0$. First note that the number 0 is a lower bound of A.
(1 point)

To show that we have found the greatest lower bound we can proceed in two different ways.

Method 1. Let ℓ be any lower bound for A, then $\ell \leq n/(n+k)$ for all $n, k \in \mathbb{N}$. In particular, by taking $n = 1$, it follows that $\ell \leq 1/(1+k)$ for all $k \in \mathbb{N}$. The sequence $x_k = 1/(1+k)$ is convergent. By the Order Limit Theorem it follows that $\ell \leq \lim x_k$ which implies that $\ell \leq 0$. This shows that $\inf A = 0$.
(5 points)

Method 2. Let $\epsilon > 0$ be arbitrary, and take $n = 1$ and $k \in \mathbb{N}$ such that $1/k < \epsilon$. Then

$$\frac{n}{n+k} = \frac{1}{k+1} < \frac{1}{k} < \epsilon.$$

This shows that ϵ is not a lower bound of A. We conclude that $\sup A = 1$.
(5 points)

Solution of problem 2 (5 + 5 + 5 = 15 points)

(a) Since $x_1^2 = 25 > 5$ the statement is true for $n = 1$.
(1 point)

Now assume that $x_n^2 > 5$ for some $n \in \mathbb{N}$, then

$$x_{n+1}^2 - 5 = \frac{1}{4}\left(x_n^2 + 10 + \frac{25}{x_n^2}\right) - \frac{20}{4} = \frac{1}{4}\left(x_n^2 - 10 + \frac{25}{x_n^2}\right) = \frac{1}{4}\left(x_n - \frac{5}{x_n}\right)^2 \geq 0.$$

Since $x_n > \sqrt{5}$, it follows that $x_{n+1}^2 - 5 > 0$. By induction, $x_n^2 > 5$ for all $n \in \mathbb{N}$.
(3 points)

Either $x_{n+1} > \sqrt{5}$ or $x_{n+1} < -\sqrt{5}$. From the definition of the sequence (x_n) it follows that $x_n > 0$ implies that $x_{n+1} > 0$. Since $x_1 > 0$, it follows that $x_n > 0$ for all $n \in \mathbb{N}$ and this rules out the possibility of having $x_n < -\sqrt{5}$.
(1 point)

(b) By part (a) it follows that

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{5}{x_n}\right) = \frac{x_n}{2} - \frac{5}{2x_n} = \frac{x_n^2 - 5}{2x_n} > 0,$$

which implies that $x_{n+1} < x_n$.
(5 points)

(c) By parts (a) and (b) it follows that (x_n) decreases and is bounded from below. The Monotone Convergence Theorem implies that $x = \lim x_n$ exists.
(3 points)

Note that $x = \lim x_{n+1}$ as well. The Algebraic Limit Theorem shows that x satisfies the equation $x = \frac{1}{2}(x + 5/x)$, or, equivalently, $x^2 = 5$. Hence, $x = \sqrt{5}$.
(2 points)

Solution of problem 3 (3 + 9 + 3 = 15 points)

(a) A point $x \in \mathbb{R}$ is a limit point of $A \subset \mathbb{R}$ if for every $\epsilon > 0$ the neighborhood $V_\epsilon(x)$ intersects A in a point other than x itself.
(3 points)

(b) *Proof of "$\Rightarrow$":* Note that $A \subset \overline{A}$. If $V_\epsilon(x)$ intersects A in a point other than x itself, then $V_\epsilon(x)$ certainly intersects $\overline{A}$ in a point other than x itself.
(2 points)

Proof of "$\Leftarrow$": If x is a limit point of $\overline{A}$, then for every $\epsilon > 0$ there exists $y \in V_\epsilon(x) \cap \overline{A}$ with $y \neq x$.
(2 points)

Note that either $y \in A$ or $y \in L$. If $y \in A$, then it follows that x is a limit point of A and we are done.
(2 points)

If $y \in L$, then for all $\delta > 0$ there exists $z \in V_\delta(y) \cap A$ with $z \neq y$. For $\delta > 0$ sufficiently small we have $V_\delta(y) \subset V_\epsilon(x) \setminus \{x\}$ and this implies that $z \neq x$. Therefore, we have shown that $V_\epsilon(x)$ intersects A in a point different from x and thus x is a limit point of A.
(3 points)

(c) From part (b) it follows that A and $\overline{A}$ have the same limit points. Hence, $\overline{A}$ by definition contains all of its limit points and hence is closed.
(3 points)

Solution of problem 4 (5 + 5 + 5 = 15 points)

(a) Define the function $g(x) = f(x) - x$ which is continuous by the Algebraic Continuity Theorem. Note that $g(0) = f(0) = 1 > 0$ and $g(3) = f(3) - 3 = -1 < 0$. By the Intermediate Value Theorem there exists $a \in (0, 3)$ such that $g(a) = 0$, or, equivalently, $f(a) = a$.
(5 points)

(b) By the Mean Value Theorem there exists $b \in (0, 3)$ such that

$$f'(b) = \frac{f(3) - f(0)}{3 - 0} = \frac{2 - 1}{3 - 0} = \frac{1}{3}.$$

(5 points)

(c) Since $f(1) = f(3)$ Rolle's Theorem implies that there exists $d \in (1, 3)$ such that $f'(d) = 0$.
(2 points)

Since $0 < \frac{1}{4} < \frac{1}{3}$ it follows by Darboux's Theorem that there exists $c \in (0, 3)$ such that $f'(c) = \frac{1}{4}$.
(3 points)

Solution of problem 5 (9 + 6 = 15 points)

(a) Since the series $\sum_{n=0}^{\infty} a_n x_0^n$ converges, it follows that $\lim a_n x_0^n = 0$. In particular, the sequence $(a_n x_0^n)$ is bounded, which means that there exists a constant $C > 0$ such that $|a_n x_0^n| \leq C$ for all $n \in \mathbb{N}$.
(3 points)

Let $x \in (-x_0, x_0)$ so that $|x| < |x_0|$. Then

$$|a_n x^n| = \left| a_n x_0^n \left(\frac{x}{x_0} \right)^n \right| = |a_n x_0^n| \cdot \left| \frac{x}{x_0} \right|^n \leq C \left| \frac{x}{x_0} \right|^n \quad \text{for all } n \in \mathbb{N}.$$

(3 points)

Since $|x/x_0| < 1$ the geometric series $\sum_{n=0}^{\infty} C|x/x_0|^n$ converges. By the Comparison Test it follows that the series $\sum_{n=0}^{\infty} |a_n x^n|$ converges.
(3 points)

(b) Let $0 < c < x_0$ and $f_n(x) = a_n x^n$. We have $|f_n(x)| \leq |a_n c^n| =: M_n$ for all $x \in [-c, c]$ and $n \in \mathbb{N} \cup \{0\}$. By part (a) it follows that $\sum_{n=0}^{\infty} M_n < \infty$. By the Weierstrass test it follows that $\sum_{n=0} f_n(x)$ converges uniformly on $[-c, c]$.
(6 points)

Solution of problem 6 (10 + 5 = 15 points)

(a) Let $\epsilon > 0$ be arbitrary and take the partition

$$P = \{x_0 = 0, x_1 = r_j - \tfrac{1}{2}\epsilon, x_2 = r_j + \tfrac{1}{2}\epsilon, x_3 = 1\}.$$

(3 points)

We have

$$\begin{aligned} m_1 &= \inf_{x\in[x_0,x_1]} f_j(x) = 0, \\ m_2 &= \inf_{x\in[x_1,x_2]} f_j(x) = 0, \\ m_3 &= \inf_{x\in[x_2,x_3]} f_j(x) = 0, \\ M_1 &= \sup_{x\in[x_0,x_1]} f_j(x) = 0, \\ M_2 &= \sup_{x\in[x_1,x_2]} f_j(x) = 1, \\ M_3 &= \sup_{x\in[x_2,x_3]} f_j(x) = 0, \end{aligned}$$

(3 points)

This gives

$$U(f_j, P) - L(f_j, P) = \sum_{k=1}^{3}(M_k - m_k)(x_k - x_{k-1}) = (r_j + \tfrac{1}{2}\epsilon) - (r_j - \tfrac{1}{2}\epsilon) = \epsilon.$$

This shows that each f_j is integrable.
(3 points)

In addition, we have that

$$\int_0^1 f_j = \sup\{L(f_j, P) \,:\, P \text{ partition of } [0,1]\} = 0.$$

(1 point)

(b) We can rewrite the function $F(x) = \sum_{j=1}^{\infty} f_j(x)$ as

$$F(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Let P be any partition of $[0, 1]$. Since each subinterval of P contains a rational number it follows that $U(F, P) = 1$.
(2 points)

Since each subinterval of P contains an irrational number it follows that $L(F, P) = 0$.
(2 points)

Therefore, F is not integrable.
(1 point)